

Two Bijections for Dyck Path Parameters

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The Motzkin number M_n ([A001006](#)) counts Motzkin n -paths: lattice paths of upsteps $U = (1, 1)$, flatsteps $F = (1, 0)$ and downsteps $D = (1, -1)$ such that (i) the path contains n steps, (ii) the number of U s = number of D s, and (iii) the path never dips below the horizontal line joining its initial and terminal points (ground level) [[1](#)]. A Dyck n -path is a Motzkin $(2n)$ -path with no flatsteps, counted by the Catalan number C_n . A UUU -free Dyck path is one that contains no consecutive UUU and so on. A descent in a path is a maximal sequence of contiguous downsteps. A short descent is one consisting of a single downstep. A terminal descent is one that ends the path. Thus a Dyck path is UDU -free iff it contains no short nonterminal descents. Each upstep in a Dyck (or Motzkin) path has a matching downstep: the first one encountered directly East of the upstep.

The first result is easy, and also appears in [[2](#)].

Theorem 1. *The number of UUU -free Dyck n -paths is M_n .*

Proof. Given a UUU -free Dyck n -path, change each UUD to U , then change each remaining UD to F . This is a bijection to Motzkin n -paths. For example with $n = 5$: $UUUDUDDDD- > UFFUDD$. The inverse is obvious. \square

The next bijection gives the distributions both of the parameter “number of $UDUs$ ” ([A091869](#)) [[3](#)] and the parameter “number of $DDUs$ ” ([A091894](#)) on Dyck paths.

Theorem 2.

(i) *The number of Dyck n -paths containing exactly k $UDUs$ is $\binom{n-1}{k} M_{n-1-k}$ (Donaghey distribution).*

(ii) *The number of Dyck n -paths containing exactly k $DDUs$ is $\binom{n-1}{2k} 2^{n-1-2k} C_k$ (Touchard distribution).*

Proof. A bicolored Motzkin n -path is a Motzkin path of length n in which each flatstep is given one of two colors, say green and black, denoted by wavy and flat overlines respectively. There is an obvious bijection to Dyck $(n+1)$ -paths: replace each U by UU , D by DD , \overline{F} by UD , \widetilde{F} by UD and then prepend a U and append a D .

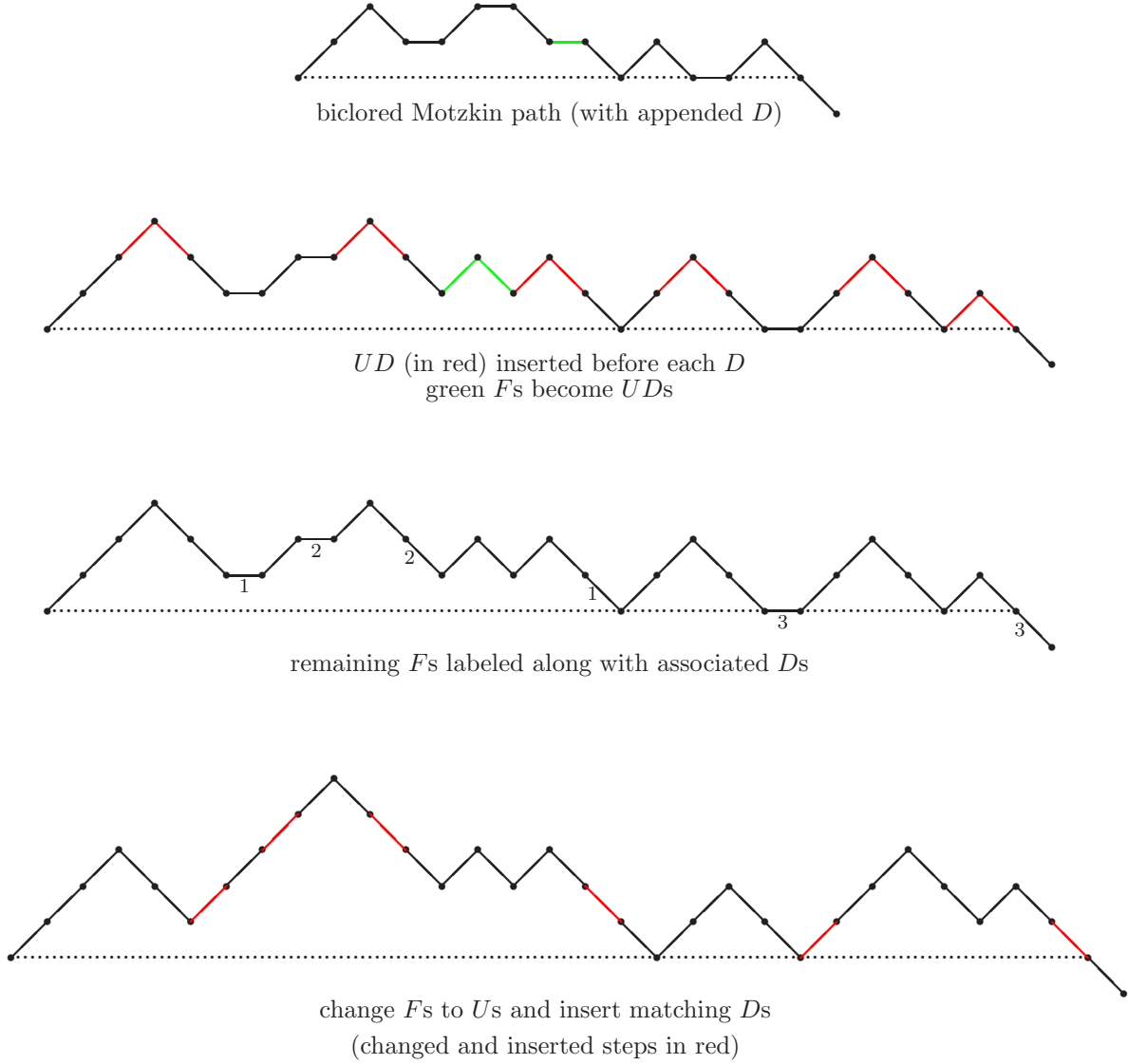
Here is a less obvious bijection that takes $\# \widetilde{F}$ s to $\# UDUs$ and $\# D$ s to $\# DDUs$. Since it is easy to count bicolored Motzkin paths either by $\# \widetilde{F}$ s or by $\# D$ s, we get the stated distributions.

Given a bicolored Motzkin n -path, first append a downstep as a convenience. Now every flatstep F has an associated downstep: the first D whose initial point is on the ray obtained by extending F eastward. Then

1. Leave U s intact
2. Replace D by UDD
3. Replace \widetilde{F} by UD
4. Replace \overline{F} by U and insert a D immediately before its associated downstep (thus the new U and inserted D will be a matching UD pair and the result is the same no matter in what order the \overline{F} s are processed).

Finally, delete the appended D (it remains undisturbed). An example with $n = 14$ is

illustrated.



The map is invertible: \tilde{F} s are recaptured as UD s followed by a U , D s are recaptured as UDD s, \overline{F} s are recaptured as U s whose matching D is in the *interior* of a descent of length ≥ 3 and the original U s are recaptured as U s such that both the U itself and its matching D are followed by a U . \square

This map restricted to Motzkin paths (no green F s) is a bijection from Motzkin n -paths to UDU -free Dyck $(n+1)$ -paths. Further, note that a Motzkin path contains no flatsteps at ground level iff the corresponding UDU -free Dyck path ends with UD . So, by deleting this UD , Motzkin n -paths with no flatsteps at ground level ([A005043](#)) correspond to Dyck n -paths with no short descents.

References

- [1] Richard P. Stanley, *Enumerative Combinatorics* Vol. 2, Cambridge University Press, 1999.
- [2] Sergi Elizalde and Toufik Mansour, Restricted Motzkin permutations, Motzkin paths, continued fractions, and Chebyshev polynomials, [preprint](#).
- [3] Emeric Deutsch and Louis W. Shapiro, A bijection between ordered trees and 2-Motzkin paths and its many consequences, *Discrete Math.* **256** (2002), 655-670.